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# Varieties of modules and $p$ -blocks of finite groups

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## 1 Introduction

Let  $G$  be a finite group and  $k$  an algebraically closed field of characteristic  $p > 0$ . For a finitely generated  $kG$ -module  $M$ , we denote the closed subvariety of  $V_G(k)$  defined by the annihilator of  $\text{Ext}_{kG}^*(M, M)$  in  $H^*(G, k)$  by  $V_G(M)$ , where  $V_G(k)$  is the maximal ideal spectrum of the cohomology ring  $H^*(G, k)$ . In this note, we consider varieties of indecomposable modules in a  $p$ -block of  $kG$ . It is known that for any homogeneous closed subvariety  $V$  of  $V_G(k)$ , there is a  $kG$ -module  $M$  such that  $V_G(M) = V$ . On the other hand, if  $M$  is an indecomposable  $kG$ -module, then the variety  $V_G(M)$  is connected (as a projective variety) ([C]). Our main result is the following.

**Theorem** *Let  $B$  be a block of  $kG$  with defect group  $D$ . Let  $V$  be a connected homogeneous closed subvariety of  $V_G(k)$ . Then  $V = V_G(L)$  for some indecomposable  $kG$ -module  $L$  in  $B$  if and only if  $V = \text{res}_{G,D}^*(W)$  for some connected homogeneous closed subvariety  $W$  of  $V_D(k)$ .*

Here, we say  $V$  is connected if it is connected as a projective variety.

In section 4, we study extensions of graded modules over a graded algebra  $A$ , which is a twisted algebra (in the sense of [Z]) of the group algebra of an elementary abelian 2-group. We state some results on the complexity  $c(M)$  and the rate of growth of  $\text{Ext}_A^*(M, M)$  for a graded  $A$ -module  $M$ . It is known that these are equal if  $A$  is a group algebra.

## 2 Varieties of modules in $B$

Let  $B$  be a block of  $kG$  and  $D$  a defect group of  $B$ . We denote by  $V_B$  the union of varieties of all finitely generated  $kG$ -modules in  $B$ . It is easy to see

that  $V_B = \text{res}_{G,D}^*(V_D(k))$ , where  $\text{res}_{G,D}^*$  is the map induced by the restriction,  $H^*(G, k) \rightarrow H^*(D, k)$ . Moreover, there is a finitely generated  $kG$ -module  $M$  in  $B$  such that  $V_G(M) = V_B$ . If  $V$  is a connected homogeneous closed subvariety of  $V_B$ , then  $V$  is not necessarily a variety of some indecomposable  $kG$ -module in  $B$ . The problem is that, in general,  $V$  does not come from a *connected* homogeneous closed subvariety of  $V_D(k)$  (see Example 2.4).

**Theorem 2.1** *Let  $H$  be a subgroup of  $G$ ,  $M$  a  $kG$ -module and  $V$  a connected homogeneous closed subvariety of  $V_G(k)$ . Suppose that the trivial  $kH$ -module is a direct summand of  $M$  as a  $kH$ -module. If  $V = \text{res}_{G,H}^*(W)$  for some connected homogeneous closed subvariety  $W$  of  $V_H(k)$ , then there exists an indecomposable  $kG$ -module  $L$  such that  $V_G(L) = V$  and  $\text{Hom}_{kG}(M, L) \neq 0$ .*

Now, we consider the varieties of  $kG$ -modules in a block  $B$ . Since the varieties of  $kG$ -modules in  $B$  is contained in  $V_B$ , we consider only such a variety.

**Corollary 2.2** *Let  $B$  be a block of  $kG$  with defect group  $D$ . Let  $V$  be a connected homogeneous closed subvariety of  $V_B$ . Then  $V = V_G(L)$  for some indecomposable  $kG$ -module  $L$  in  $B$  if and only if  $V = \text{res}_{G,D}^*(W)$  for some connected homogeneous closed subvariety  $W$  of  $V_D(k)$ .*

*Proof.* Suppose that there exists an indecomposable  $kG$ -module  $L$  in  $B$  such that  $V_G(L) = V$ . Then there exists an indecomposable  $kD$ -module  $N$  such that  $L|N \uparrow^G$  and  $N|L \downarrow_D$ . So we have that  $\text{res}_{G,D}^*(V_D(N)) = V$ . Conversely, suppose that there exists a connected homogeneous closed subvariety  $W$  of  $V_D(k)$  such that  $\text{res}_{G,D}^*(W) = V$ . Note that there exists a  $kG$ -module  $M$  in  $B$  such that  $k|M \downarrow_D$ . By Theorem 2.1, there exists an indecomposable  $kG$ -module  $L$  such that  $V_G(L) = V$  and  $\text{Hom}_{kG}(M, L) \neq 0$ . In particular,  $L$  belongs to  $B$ .

Let  $V$  be a connected homogeneous closed subvariety of  $V_G(k)$ . If  $H$  is a Sylow  $p$ -subgroup of  $G$ , then it is easy to see that  $V = \text{res}_{G,H}^*(W)$  for some connected homogeneous closed subvariety  $W$  of  $V_H(k)$ . So we have,

**Corollary 2.3** ([H1]) *Let  $V$  be a connected homogeneous closed subvariety of  $V_G(k)$ . Then there exists an indecomposable  $kG$ -module  $L$  such that  $V_G(L) = V$  and  $\text{Hom}_{kG}(k, L) \neq 0$ .*

**Example 2.4** Let  $p = 2$ . Let  $G$  be a 2-nilpotent group generated by

$$x_i, y_i, z_i, u, v \quad (i = 1, 2)$$

with relations,

$$x_i^2 = y_i^3 = z_i^2 = u^2 = v^2 = 1, \quad y_i^{x_i} = y_i^{-1},$$

$$\begin{aligned}
x_i^v &= x_j, \quad y_i^v = y_j, \quad z_i^v = z_j, \\
[a_i, b_j] &= [a_i, z_i] = [a_i, u] = [v, u] = 1, \\
(a, b &= x, y, z, \quad 1 \leq i \neq j \leq 2).
\end{aligned}$$

So  $G \cong ((S_3 \times C_2) \wr C_2) \times C_2$ , where we denote the symmetric group of degree 3 by  $S_3$  and a cyclic group of order 2 by  $C_2$ . We set

$$D = \langle x_2, z_1, z_2, u \rangle, \quad E = \langle x_2 u, z_1 \rangle, \quad F = \langle x_2, z_2 \rangle,$$

$$V = \text{res}_{G,E}^*(V_E(k)) \cup \text{res}_{G,F}^*(V_F(k)).$$

Then  $G$  has a 2-block  $B$  with defect group  $D$ . Moreover,  $V$  is a connected homogeneous closed subvariety of  $V_B$ . But there exists no connected homogeneous closed subvariety  $W$  of  $V_D(k)$  such that  $V = \text{res}_{G,D}^*(W)$ . So there exists no indecomposable  $kG$ -module  $L$  in  $B$  such that  $V_G(L) = V$ .

### 3 Some associated primes in $H^*(G, k)$

Let  $G$  be a  $p$ -group. The complexity  $c(M)$  of a finitely generated  $kG$ -module  $M$  is the smallest nonnegative integer  $c$  such that

$$\lim_{n \rightarrow \infty} \frac{\dim_k \Omega^n(M)}{n^c} = 0.$$

It is known that  $c(M) = \dim V_G(M) = \dim H^*(G, k)/I(M)$  where  $I(M)$  is the annihilator of  $H^*(G, M)$  in  $H^*(G, k)$  ([B, Chapter 5]). So there exists a minimal associated prime  $P$  of  $H^*(G, M)$  such that  $\dim H^*(G, k)/P = c(M)$  ([M, Theorem 6.5]). Since  $P$  is an associated prime ideal, there exists a homogeneous element  $x \in H^*(G, M)$  such that  $P = \text{ann } x$ . In particular,  $\dim H^*(G, M)/\text{ann } x = c(M)$ .

**Definition** Let  $G$  be a  $p$ -group and  $M$  a finitely generated  $kG$ -module. Suppose that  $1 \leq i \leq c(M)$ . We define  $m_i(M)$  to be the smallest integer  $m \geq 0$  such that  $\dim H^*(G, k)/\text{ann } x \geq i$  for some  $x \in H^m(G, M)$ . Then we have

$$m_1(M) \leq m_2(M) \leq \cdots \leq m_{c(M)}(M) < \infty$$

by the above argument.

**Example 3.1** (1) Let  $p = 2$  and  $G = C_2 \times C_2$ . If  $M$  is a nonprojective indecomposable  $kG$ -module, then  $M$  is either periodic or isomorphic to  $\Omega^n(k)$  for some  $n \in \mathbb{Z}$ . If  $M$  is periodic, then  $m_1(M) = 0$ . On the other hand, we

have  $m_2(\Omega^n(k)) = \max\{n, 0\}$ .

(2) Let  $p = 2$  and  $G = C_2 \times C_2 \times C_2$ . Fix any positive integer  $n$ . Then,

$$\sup\{m_2(M) | M : \text{f.g. } kG\text{-module}, c(M) = 3, \dim_k M \leq n\} < \infty.$$

**Question 3.2** Let  $G$  be a  $p$ -group. Fix  $n, i > 0$ . Then,

$$\sup\{m_i(M) | M : \text{f.g. } kG\text{-module}, i \leq c(M), \dim_k M \leq n\} < \infty ?$$

## 4 Extensions of modules over some graded algebras

In this section, we assume that  $p = 2$ . Let

$$A = k \langle x_1, \dots, x_r \rangle / (x_i^2, a_i x_i x_j + a_j x_j x_i, 1 \leq i, j \leq r)$$

for  $a_i \in k, a_i \neq 0$ . Then  $A$  is a finite dimensional local selfinjective graded  $k$ -algebra with  $\deg x_i = 1$  (see [H2], [Z] for more details). For a finitely generated  $A$ -module  $M$ , we define the rate of growth  $\gamma(\text{Ext}_A^*(M, M))$  of  $\text{Ext}_A^*(M, M)$  to be the smallest nonnegative integer  $s$  such that

$$\lim_{n \rightarrow \infty} \frac{\dim_k \text{Ext}_A^n(M, M)}{n^s} = 0.$$

Then,  $0 \leq \gamma(\text{Ext}_A^*(M, M)) \leq c(M) \leq r$ .

**Theorem 4.1** Let  $M$  be a finitely generated graded  $A$ -module. If  $\gamma(\text{Ext}_A^*(M, M)) = 0$ , then  $c(M) \leq r/2$ .

**Remark** (1) If  $a_i = 1$  for every  $i$ , then  $A$  is a group algebra of an elementary abelian 2-groups. In this case,  $\gamma(\text{Ext}_A^*(M, M)) = c(M)$ .

(2) ([H2]) If we take  $a_1, \dots, a_r$  suitably, then  $A$  satisfies the following.

(\*) For any  $1 \leq s \leq r/2$ , there exists a graded  $A$ -module  $M$  such that  $c(M) = s$  and  $\gamma(\text{Ext}_A^*(M, M)) = 0$ .

Suppose that  $r = 3$ . If  $M$  is a graded  $A$ -module and  $c(M) = 3$ , then  $\gamma(\text{Ext}_A^*(M, M)) \geq 1$  by Theorem 4.1. Using Example 3.1(2), we can prove the following.

**Proposition 4.2** Suppose that  $r = 3$ . If  $M$  is a graded  $A$ -module with

$c(M) = 3$ , then  $\gamma(\text{Ext}_A^*(M, M)) \geq 2$ .

**Question 4.3** Suppose that  $r = 3$ . If  $M$  is a graded  $A$ -module with  $c(M) = 3$ , then  $\gamma(\text{Ext}_A^*(M, M)) = 3$  ?

Suppose that Example 3.1(2) is true if we replace  $m_2(M)$  by  $m_3(M)$ . Then the equality in Question 4.3 holds.

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